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DYNAMICS OF ISOTHERMAL FLUIDS IN SPECIAL RELATIVITY

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ABSTRACT. Following LeFloch and Shelukhin who considered the non-relativistic setting, we investigate here the Cauchy problem for the relativistic Euler equations. We establish the existence of globally defined, bounded measurable, entropy solutions with arbitrary large amplitude; the mass density of the fluid may vanish and the velocity field approach the speed of light. The Euler equations become strongly degenerate in both regimes as the conservative and flux variables either vanish or blow-up.

1. INTRODUCTION

We review recent results by the authors on the relativistic Euler equations and give an outline of the proof. A complete proof will appear in our forthcoming paper [8]. The principal fluid unknowns consist of the co-moving mass density $\rho \geq 0$ and the velocity field $v \in (-c, c)$, where c denotes the speed of light. We are interested in weak solutions containing shock waves, with arbitrary large amplitude. We consider here isothermal fluids, governed by the linear pressure law $p(\rho) = k^2 \rho$, where $k > 0$ is the (local, constant) sound speed satisfying $k < c$. For the Cauchy problem, Smoller and Temple [13] established the existence of a globally defined, entropy solution, under the assumption that the initial density is bounded, is bounded away from zero, and has bounded variation. Their result took its root in Nishida's earlier work for the non-relativistic version of this system [12].

Our aim here is to encompass more general solutions, which may contain vacuum states ($\rho = 0$) and high-velocities. Such solutions arise naturally in physical applications: for instance, a star is described by a compactly supported mass density function, and the boundary between the vacuum exterior region and the interior of the star must be coped with. We observe that, at points where ρ vanishes, the Euler equations are highly degenerate since the conservative variables vanish identically and the velocity field can not be uniquely determined. Another closely related singularity is the limit when the fluid velocity approaches the light speed and the relativistic Euler equations fail to be uniformly strictly hyperbolic, as the wave speeds coincide asymptotically. These features lead to very challenging mathematical questions concerning the existence and the behavior of entropy solutions of the Euler equations.

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We generalize here to relativistic fluids the existence results in LeFloch and Shelukhin [7]. Recall that the equations under consideration read

$$\partial_t \left(\frac{\rho\, c^2 + p\, v^2/c^2}{c^2 - v^2} \right) + \partial_x \left(\frac{(p + \rho c^2)\, v}{c^2 - v^2} \right) = 0,$$
$$\partial_t \left(\frac{(p + \rho\, c^2)\, v}{c^2 - v^2} \right) + \partial_x \left(\frac{(p + \rho\, v^2)\, c^2}{c^2 - v^2} \right) = 0,$$

which, by setting $\varepsilon = 1/c$ and using the condition $p(\rho) = k^2\, \rho$, transform into

$$\begin{aligned} &\partial_t \left(\frac{1 + \varepsilon^4\, k^2\, v^2}{1 - \varepsilon^2\, v^2}\, \rho \right) + \partial_x \left(\frac{1 + \varepsilon^2\, k^2}{1 - \varepsilon^2\, v^2}\, \rho\, v \right) = 0, \\ &\partial_t \left(\frac{1 + \varepsilon^2\, k^2}{1 - \varepsilon^2\, v^2}\, \rho\, v \right) + \partial_x \left(\frac{k^2 + v^2}{1 - \varepsilon^2\, v^2}\, \rho \right) = 0. \end{aligned} \tag{1.1}$$

In the limit $\varepsilon = 0$, this is nothing but the non-relativistic Euler system

$$\begin{aligned} &\partial_t \rho + \partial_x\, (\rho\, v) = 0, \\ &\partial_t\, (\rho\, v) + \partial_x\, ((k^2 + v^2)\, \rho) = 0, \end{aligned} \tag{1.2}$$

studied earlier in [7]. Another technique for the existence theory to (1.2) was also developed by Huang and Wang [5]. The present work restricts attention to isothermal fluids and solutions containing vacuum states; we do not try to review here the literature on non-iseothermal fluids and other physical settings for which we refer to [9] and the references therein. For background on the relativistic Euler equations we refer the reader to [4, 10, 14].

2. Main result

We are typically interested in the Cauchy problem with data prescribed on the initial line $t = 0$. As is usual, we seek for weak solutions (in the sense of distributions) satisfying entropy inequalities which are classically imposed for the sake of uniqueness.

A Lipschitz continuous map $(\mathcal{U}, \mathcal{F})$ is called an entropy pair if every smooth solution of (1.1) satisfies

$$\partial_t \mathcal{U}(\rho, v) + \partial_x \mathcal{F}(\rho, v) = 0.$$

We will be interested in weak entropy pairs which by definition, vanish on the vacuum. For instance, (g, h) and (h, f) are (trivial) weak entropy pairs. Define

$$\varepsilon' := \frac{2\varepsilon}{1 + \varepsilon^2}.$$

Definition 2.1. A *tame region* is a set of the form

$$\mathcal{T}_\varepsilon(M) := \{\rho, v \,/\, 0 \le \rho \le M, \quad 1 - \varepsilon |v| \ge (\rho/M)^{\varepsilon'}\} \tag{2.1}$$

for some constant $M > 0$. A pair of measurable and bounded functions ρ_0, v_0: $\mathbb{R} \to \mathbb{R}$ is a *tame initial data* if its range is included in a tame region.

Given a tame initial data ρ_0, v_0, a pair of measurable and bounded functions ρ, v: $\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is called a tame entropy solution to the isothermal relativistic Euler equations

(1.1) associated with the initial data ρ_0, v_0, if its range ρ, v is included in a tame region and, moreover,

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \Big(\mathcal{U}(\rho,v)\,\theta_t + \mathcal{F}(\rho,v)\,\theta_x\Big)\,dxdt + \int_{\mathbb{R}} \mathcal{U}(\rho_0,v_0)\,\theta(0,\cdot)\,dx \geq 0 \tag{2.2}$$

for every convex, smooth, weak entropy pair $(\mathcal{U},\mathcal{F})$ of the isothermal relativistic Euler equations and every test-function θ supported in $[0,\infty)\times\mathbb{R}$.

Observe that the inequality (2.1) shows that the velocity can approach the speed of light (normalized here to be $1/\varepsilon$) only if the density is also approaching zero. Such a condition is natural and is equivalent to certain bounds on the Riemann invariants. Clearly, the entropy pairs under consideration need only be Lipschitz continuous within any tame region, as those are the only relevant regions in our definition of entropy solution.

Our main result is:

Theorem 2.2. *Given $\varepsilon \in (0,1)$ and a tame initial data ρ_0, v_0 there exists a tame entropy solution ρ, $v\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ to the relativistic Euler equations for isothermal fluids* (1.1) *associated with ρ_0, v_0.*

All of the estimates under consideration here are uniform with respect to $\varepsilon \to 0$ and, therefore, we can recover immediately from Theorem 2.2 the existence result obtained earlier in [7]. Our approach will rely on the following observation, which was already essential in [7] and is true also for the relativistic equations: If (ρ, v) is a (weak, entropy) solution of the relativistic Euler equations for isothermal fluid (1.1), then for every positive constant λ, the function $(\lambda\rho, v)$ is also a (weak, entropy) solution of the same equations.

Our general strategy of proof follows on one hand DiPerna [2, 3], who obtained *bounded solutions* $\rho \geq 0$ and $v \in \mathbb{R}$ for non-relativistic fluids satisfying $p(\rho) = k^2\rho^\gamma$, with $\gamma > 1$ (polytropic gas), and on the other hand LeFloch and Shelukhin [7], who extended DiPerna's analysis to include isothermal fluids satisfying $p(\rho) = k^2\,\rho$ and observed that the velocity field v need not be bounded.

As was observed earlier, the wave speed are independent of the density variable. The main difficulty for the analysis lies in the lack of uniform strict hyperbolicity of the system when the fluid velocity approaches the light speed. Our analysis will proceed along the following lines:

- *Mathematical entropy pairs.* Our first task is to construct a sufficiently large family of entropy pairs. This problem is reduced to a linear hyperbolic equation in ρ, v, which can be (formally) solved by introducing the corresponding Riemann function $\mathcal{R}$ and the entropy kernel χ. In turn, the entropy pairs can be expressed by an explicit formula in terms of χ. Contrary to the case of polytropic fluids studied in [2, 3], data for the entropy kernel should be imposed away from the vacuum; we prescribe an initial data on $\rho = 1$ as in [7].
- *Singularities of the entropy kernel.* The function χ is discontinuous along the boundary $\partial\mathcal{K}$ of its support $\mathcal{K}$, so that its first-order derivatives exhibit Dirac masses. In the case

$\varepsilon = 0$ studied in [7], the kernel was given by an explicit formula. In contrast, when $\varepsilon > 0$ we need to provide uniform estimates on χ and, in particular, we determine explicitly the traces of its first-order derivatives along $\partial\mathcal{K}$.

- *A priori bounds.* We derive a priori bounds on solutions generated by the Lax-Friedrichs scheme. Our bounds are uniform in ε and show that the density remains bounded while the tame condition between the density and the velocity is exhibited.
- *Reduction of the Young measure.* The final step consists of a technical lemma concerning the support of the Young measure $\nu = \nu_{t,x}$ associated with the sequence ρ^h, v^h. We follow closely the analysis given in [7] in the case $\varepsilon = 0$. The key term of interest contains a key coefficient denoted below by $D(\rho)$, which can be made to be bounded way from zero, by taking advantage of the scaling invariance. We apply the so-called div-curl lemma (see [11, 15]) and derive Tartar's commutation relations established in [15]. The objective is to prove that the measure ν is a point mass measure by using a sufficiently large class of entropy pairs. We need to handle products of BV functions with bounded variation and measures, following the theory in Dal Maso, LeFloch and Murat [1].

3. Some basic properties

Scaling properties of (1.1) will play an important role. Observe that the scaling

$$v' = v/k, \quad t' = k\,t, \quad \varepsilon' = k\,\varepsilon$$

allows one to reduce the system to the same system with $k = 1$. In view of the physical constraint $0 < k < c$ between the sound speed and the light speed this amounts to impose $c > 1$. The limiting case $c \to 1$ corresponds to the special case where the sound speed and the light speed coincide. From now on, we suppose that $k = 1$, i.e.

$$\begin{aligned} &\partial_t \left(\frac{1 + \varepsilon^4 v^2}{1 - \varepsilon^2 v^2}\,\rho\right) + \partial_x \left(\frac{1 + \varepsilon^2}{1 - \varepsilon^2 v^2}\,\rho\, v\right) = 0, \\ &\partial_t \left(\frac{1 + \varepsilon^2}{1 - \varepsilon^2 v^2}\,\rho\, v\right) + \partial_x \left(\frac{1 + v^2}{1 - \varepsilon^2 v^2}\,\rho\right) = 0. \end{aligned} \tag{3.1}$$

The velocity is restricted to lie in the interval $(-1/\varepsilon, 1/\varepsilon)$; note that the conservative and flux variables blow-up when $v \to \pm 1/\varepsilon$. The range of physical interest for ε is

$$0 < \varepsilon < 1,$$

the limiting case $\varepsilon = 0$ and $\varepsilon = 1$ corresponding to the non-relativistic model (speed of light infinite) and the scalar field model (the sound speed and the light speed coincide), respectively. The latter system reads

$$\begin{aligned} &\partial_t \left(\frac{1 + v^2}{1 - v^2}\,\rho\right) + \partial_x \left(\frac{2}{1 - v^2}\,\rho\, v\right) = 0, \\ &\partial_t \left(\frac{2}{1 - v^2}\,\rho\, v\right) + \partial_x \left(\frac{1 + v^2}{1 - v^2}\,\rho\right) = 0, \end{aligned} \tag{3.2}$$

which is simply equivalent to the linear wave equation; this is clear by introducing the change of variable $a := \dfrac{1+v^2}{1-v^2}\,\rho$ and $b := \dfrac{2}{1-v^2}\,\rho\, v$, thus

$$\partial_t a + \partial_x b = 0, \quad \partial_t b + \partial_x a = 0. \tag{3.3}$$

Lemma 3.1. *When expressed as a system of two conservation laws, the eigenvalues of the corresponding Jacobian matrix of* (3.1) *are found to be*

$$\lambda_1 := \frac{v-1}{1-\varepsilon^2\, v}, \qquad \lambda_2 := \frac{v+1}{1+\varepsilon^2\, v}.$$

Proof. We follow the argument due to Smoller and Temple [13]. The equations (3.1) form a nonlinear hyperbolic system of two conservation laws,

$$\partial_t G + \partial_x H = 0, \qquad \partial_t H + \partial_x F = 0, \tag{3.4}$$

where

$$G(\rho,v) = \frac{1+\varepsilon^4 v^2}{1-\varepsilon^2 v^2}\rho, \quad H(\rho,v) = \frac{1+\varepsilon^2}{1-\varepsilon^2 v^2}\rho v, \quad F(\rho,v) = \frac{1+v^2}{1-\varepsilon^2 v^2}\rho. \tag{3.5}$$

We recall a classical result:

Lemma 3.2. *For a given state (ρ_L, v_L), assume that the characteristic speed $\lambda_i(\rho_L, v_L)$ is a simple eigenvalue of the Jacobian matrix. Then there is a C^2-curve $(\rho, v) = K_i(\tau)$ in a state space, called the i-shock curve through (ρ_L, v_L), and a C^2-function $s = s_i(\tau)$, both defined for τ in some neighborhood of* 0 *with the following property: A state (ρ, v) can be joined to (ρ_L, v_L) by a weak i-shock of speed s if and only if $(\rho, v) = K_i(\tau)$, for some τ. Furthermore, $K_i(0) = (\rho_L, v_L)$ and*

$$s_i(0) = \lambda_i(\rho_L, v_L), \quad s_i(\tau)[G] = [H], \quad s_i(\tau)[H] = [F],$$

where, for example, $[G] = G(\rho, v) - G(\rho_L, v_L)$.

For fixed (ρ_L, v_L), the state $(\rho, v) = (\rho_R, v_R)$ lies on the Hugoniot locus through (ρ_L, v_L) if and only if the Rankine-Hugoniot condition is satisfied i.e.

$$[H]^2 = [G]\,[F]\,. \tag{3.6}$$

Due tu the configuration of the Hugoniot locus, we can assume that v is a function of ρ if $|(\rho, v) - (\rho_l, v_L)| \ll 1$. Differentiating (3.6) in ρ, we obtain

$$2[H]H' = [G]F' + G'[F] \tag{3.7}$$

where $'$ denotes $\dfrac{d}{d\rho}$. Owing de l'Hospital's rule, when $\rho \to \rho_L$,

$$H'^2 = G'F' \text{ at } \rho = \rho_L. \tag{3.8}$$

From (3.5), denoting $e = \dfrac{\varepsilon v^2}{1-\varepsilon v^2}$, we obtain $\dfrac{de}{dv} = \dfrac{2\varepsilon v}{(1-\varepsilon v^2)^2}$,

$$G' = \left[\left(\frac{(1+\varepsilon)\varepsilon v^2}{1-\varepsilon v^2} + 1\right)\rho\right]' = [((1+\varepsilon)e + 1)\,\rho]' = (1+\varepsilon)e + (1+\varepsilon)\rho\frac{de}{dv}v' + 1,$$

$$H' = \left[\frac{1+\varepsilon}{\varepsilon v}\rho e\right]' = \left[\frac{1+\varepsilon}{\varepsilon}\rho\frac{e}{v}\right]' = \frac{1+\varepsilon}{\varepsilon}\frac{e}{v} + \frac{1+\varepsilon}{\varepsilon}\rho\frac{\left(v\dfrac{de}{dv} - e\right)v'}{v^2},$$

$$F' = \left[\frac{1+\varepsilon}{\varepsilon}\rho e + \rho\right]' = \frac{1+\varepsilon}{\varepsilon}e + \frac{1+\varepsilon}{\varepsilon}\rho\frac{de}{dv}v' + 1.$$

Substituting these relations into (3.8), we find

$$0 = (v')^2\left(\frac{(1+\varepsilon)^2}{\varepsilon^2}\rho^2\frac{\left(v\dfrac{de}{dv} - e\right)^2}{v^4} - \frac{(1+\varepsilon)^2}{\varepsilon}\rho^2\left(\frac{de}{dv}\right)^2\right)$$
$$+ v'\left(\frac{2}{v^3}\frac{(1+\varepsilon)^2}{\varepsilon^2}\rho e\left(v\frac{de}{dv} - e\right) - \frac{(1+\varepsilon)^2}{\varepsilon}\rho\frac{de}{dv}(2e+1)\right)$$
$$+ \left(\left(\frac{1+\varepsilon}{\varepsilon}\frac{e}{v}\right)^2 - \{(1+\varepsilon)e + 1\}\left(\frac{1+\varepsilon}{\varepsilon}e + 1\right)\right).$$

On the above equation, the coefficient of $(v')^2 = \left(\dfrac{1+\varepsilon}{1-\varepsilon v^2}\rho\right)^2$, the coefficient of $(v')^1 = 0$ and the coefficient of $(v')^0 = -1$. Therefore we have

$$\frac{1+\varepsilon}{1-\varepsilon v^2}\rho v' = \pm 1 \text{ i.e. } v' = \pm\frac{1-\varepsilon v^2}{1+\varepsilon}\frac{1}{\rho} = \pm\frac{\varepsilon v^2}{1+\varepsilon}\frac{1}{e}\frac{1}{v}. \tag{3.9}$$

In view of the classical result cited above, we have $\lambda_i = \lim\limits_{\rho\to\rho_L}\dfrac{[G]}{[F]}$ and hence at $\rho = \rho_L$,

$$\lambda_i = \frac{\dfrac{1+\varepsilon}{\varepsilon}\dfrac{e}{v} + \dfrac{1+\varepsilon}{\varepsilon}\rho\dfrac{\left(v\dfrac{de}{dv} - e\right)v'}{v^2}}{(1+\varepsilon)e + (1+\varepsilon)\rho\dfrac{de}{dv}v' + 1} = \frac{\dfrac{1+\varepsilon}{\varepsilon}\dfrac{e}{v} \pm \left(\dfrac{v}{e}\dfrac{de}{dv} - 1\right)}{(1+\varepsilon)e \pm \dfrac{\varepsilon v^2}{e}\dfrac{de}{dv} + 1},$$

so

$$\lambda_i = \frac{(1+\varepsilon)\dfrac{v}{1-\varepsilon c^2} \pm \dfrac{1+\varepsilon v^2}{1-\varepsilon v^2}}{\dfrac{1+\varepsilon}{1-\varepsilon v^2} - \varepsilon \pm \dfrac{2\varepsilon v}{1-\varepsilon v^2}} = \frac{v-1}{1-\varepsilon v}, \frac{v+1}{1+\varepsilon v}.$$

□

Observe that the characteristic speeds are well-behaved in the closed interval $v \in [-1/\varepsilon, 1/\varepsilon]$. In particular, the relation

$$\lambda_2 - \lambda_1 = 2\frac{1-\varepsilon^2 v^2}{1-\varepsilon^4 v^2} > 0$$

shows that:

Lemma 3.3 (Hyperbolicity properties). *The Euler equations for isothermal, relativistic fluids are strictly hyperbolic in the region $|v| < 1/\varepsilon$ for all $\rho \geq 0$ (i.e. even in presence of* *vacuum singularities), but strict hyperbolicity fails at* $v = \pm 1/\varepsilon$ *(i.e. in presence of light speed singularities).*

In contrast, for polytropic perfect fluids ($p = k^2 \rho^\gamma/\gamma$ with $\gamma > 1$), the Euler equations also fail to be strictly hyperbolic at the vacuum $\rho = 0$.

Let us introduce the following two *Riemann invariants* (which are unique up to the composition by one-to-one maps)

$$
\begin{aligned}
w &:= u + R, \quad R = \frac{w - z}{2},\\
z &:= u - R, \quad u = \frac{w + z}{2},
\end{aligned}
$$

where R, u are the functions of ρ, v given by

$$
\begin{aligned}
R = \overline{R}(\rho) &:= \frac{1}{1 + \varepsilon^2} \ln \rho,\\
u = \overline{u}(v) &:= \frac{1}{2\varepsilon} \ln\Big(\frac{1 + \varepsilon v}{1 - \varepsilon v}\Big),
\end{aligned}
$$

respectively. The Riemann invariants provide a change of variables $(\rho, v) \mapsto (w, z)$ which will be often used here.

Clearly, the mapping $v \mapsto \overline{u}(v)$ is one-to-one from the bounded interval $(-1/\varepsilon, 1/\varepsilon)$ onto the real line $\mathbb{R}$. The mapping $\rho \mapsto R(\rho)$ is one-to-one from $(0, \infty)$ onto $\mathbb{R}$. It is not difficult to check that, in terms of the variables w, z, the system (3.1) takes the diagonal form

$$
\partial_t w + \lambda_2\, \partial_x w = 0, \qquad \partial_t z + \lambda_1\, \partial_x z = 0.
$$

Observe that

$$
\lambda_1(w, z) = -\frac{1}{\varepsilon}\frac{1 + \varepsilon - (1 - \varepsilon)e^{\varepsilon w}}{1 + \varepsilon + (1 - \varepsilon)e^{\varepsilon w}}, \qquad \lambda_2(w, z) = -\frac{1}{\varepsilon}\frac{1 - \varepsilon - (1 + \varepsilon)e^{\varepsilon w}}{1 - \varepsilon + (1 + \varepsilon)e^{\varepsilon w}}. \tag{3.10}
$$

Sometimes, we will also use of the "modified" Riemann invariants defined as

$$
\begin{aligned}
W &:= e^{w} = \rho^{1/(1+\varepsilon^2)} \Big(\frac{1 + \varepsilon v}{1 - \varepsilon v}\Big)^{1/(2\varepsilon)}\\
Z &= e^{-z} = \rho^{1/(1+\varepsilon^2)} \Big(\frac{1 + \varepsilon v}{1 - \varepsilon v}\Big)^{-1/(2\varepsilon)}.
\end{aligned} \tag{3.11}
$$

Note that $\rho = (WZ)^{(1+\varepsilon^2)/2}$.

Expressing the physical variables ρ, v as functions of the Riemann invariants,

$$
\rho = \overline{R}^{-1}\left((w - z)/2\right),
$$

and

$$
v = \overline{v}(w + z) := \frac{1}{\varepsilon}\frac{e^{(w+z)/c} - 1}{e^{(w+z)/c} + 1} = \frac{1}{\varepsilon}\Big(1 - \frac{2}{e^{\varepsilon(w+z)} + 1}\Big) = \frac{1}{\varepsilon}\tanh\Big(\varepsilon\frac{w + z}{2}\Big),
$$

we obtain

$$
v_w = v_z = \frac{1}{2}\Big(1 - \varepsilon^2 v^2\Big), \qquad \rho_w = -\rho_z = \frac{1}{2\,R_\rho} = \frac{1 + \varepsilon^2}{2}\rho.
$$

The derivatives of the Riemann invariants considered as functions of (ρ, v) are

$$w_\rho = -z_\rho = \frac{1}{1+\varepsilon^2}\frac{1}{\rho}, \qquad w_v = z_v = \frac{1}{1-\varepsilon^2 v^2}.$$

The corresponding eigenvectors are

$$r_1 := \left(\frac{-1}{1-\varepsilon^2 v^2}, \frac{1}{1+\varepsilon^2}\frac{1}{\rho}\right), \qquad r_2 = \left(\frac{1}{1-\varepsilon^2 v^2}, \frac{1}{1+\varepsilon^2}\frac{1}{\rho}\right).$$

The derivatives of the eigenvalues considered as functions of (ρ, v) are

$$\lambda_{1v} = \frac{1-\varepsilon^2}{(1-\varepsilon^2 v)^2}, \qquad \lambda_{2v} = \frac{1-\varepsilon^2}{(1+\varepsilon^2 v)^2},$$

and clearly $\lambda_{1\rho} = \lambda_{2\rho} = 0$. Their derivatives along the characteristic fields are

$$\begin{aligned}\nabla\lambda_1 \cdot r_1 &= \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{\rho(1-\varepsilon v)^2} > 0\\ \nabla\lambda_2 \cdot r_2 &= \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{\rho(1+\varepsilon v)^2} > 0.\end{aligned} \tag{3.12}$$

while in terms of the Riemann invariants we have

$$\begin{aligned}\lambda_{1w} &= \lambda_{1v}v_w + \lambda_{1\rho}\rho_w = \frac{(1-\varepsilon^2)(1-\varepsilon^2 v^2)}{2\,(1-\varepsilon^2 v)^2} = \lambda_{1z},\\ \lambda_{2z} &= \lambda_{2v}v_z + \lambda_{2\rho}\rho_z = \frac{(1-\varepsilon^2)(1-\varepsilon^2 v^2)}{2\,(1+\varepsilon^2 v)^2} = \lambda_{2w}.\end{aligned} \tag{3.13}$$

This shows that:

Lemma 3.4 (Genuine nonlinearity). *The Euler equations for isothermal, relativistic fluids has two genuinely nonlinear characteristic fields, in the range $|v| < 1/\varepsilon$ for all $\rho \geq 0$ (i.e. even in presence of vacuum singularities), but genuine nonlinearity fails at $\varepsilon = 1$ (i.e. in coincidence of the sound speed and the light speed).*

For latter use we provide the form of the Euler equations in the variables (ρ, v):

$$\frac{D(G,H)}{D(\rho,v)} = \begin{pmatrix} \dfrac{1+\varepsilon^4 v^2}{1-\varepsilon^2 v^2} & \dfrac{2\varepsilon^2(1+\varepsilon^2)\rho v}{(1-\varepsilon^2 v^2)^2} \\ \dfrac{(1+\varepsilon^2)v}{1-\varepsilon^2 v^2} & \dfrac{(1+\varepsilon^2)\rho(1+\varepsilon^2 v^2)}{(1-\varepsilon^2 v^2)^2} \end{pmatrix},$$

$$\frac{D(H,F)}{D(\rho,v)} = \begin{pmatrix} \dfrac{(1+\varepsilon^2)v}{1-\varepsilon^2 v^2} & \dfrac{(1+\varepsilon^2)\rho(1+\varepsilon^2 v^2)}{(1-\varepsilon^2 v^2)^2} \\ \dfrac{1+v^2}{1-\varepsilon^2 v^2} & \dfrac{2(1+\varepsilon^2)\rho v}{(1-\varepsilon^2 v^2)^2} \end{pmatrix},$$

so that we can rewrite (3.1) in the form

$$\partial_t \widetilde{u} + \partial_x \widetilde{G}(\widetilde{u}) = 0 \tag{3.14}$$

with $\widetilde{u} := \begin{pmatrix} \rho \\ v \end{pmatrix}$ and

$$D\widetilde{G}(\widetilde{u}) = (D(G,H))^{-1}D(H,F) = \begin{pmatrix} \dfrac{(1-\varepsilon^2)v}{1-\varepsilon^4 v^2} & \dfrac{(1+\varepsilon^2)\rho}{1-\varepsilon^4 v^2} \\ \dfrac{(1-\varepsilon^2 v^2)^2}{(1+\varepsilon^2)\rho(1-\varepsilon^4 v^2)} & \dfrac{(1-\varepsilon^2)v}{1-\varepsilon^4 v^2} \end{pmatrix}. \tag{3.15}$$

4. Mathematical entropies

4.1. **Entropy equation.** From the equations (1.1) one can derive additional conservation laws, which will play a central role in our analysis. By definition, a pair of mathematical entropy $\mathcal{U} = \mathcal{U}(\rho, v)$ and entropy-flux $F = F(\rho, v)$ provides a conservation law satisfied by all smooth solutions of (1.1). Setting

$$\mathcal{U}_\star = \frac{(1+\varepsilon^4 v^2)\rho}{1-\varepsilon^2 v^2}, \qquad \mathcal{F}_\star = \frac{(1+\varepsilon^2)\rho v}{1-\varepsilon^2 v^2},$$

we find

$$\nabla\mathcal{U}_\star = \left(\frac{1+\varepsilon^4 v^2}{1-\varepsilon^2 v^2}, \frac{2\varepsilon^2(1+\varepsilon^2)\rho v}{(1-\varepsilon^2 v^2)^2}\right), \qquad \nabla\mathcal{F}_\star = \left(\frac{(1+\varepsilon^2)v}{1-\varepsilon^2 v^2}, \frac{(1+\varepsilon^2)\rho(1+\varepsilon^2 v^2)}{(1-\varepsilon^2 v^2)^2}\right).$$

From (3.15) it follows that $\nabla\mathcal{F}_\star = \nabla\mathcal{U}_\star \cdot D\mathcal{F}(u)$, hence $(\mathcal{U}_\star, \mathcal{F}_\star)$ is an entropy pair. Note that, as $\varepsilon \to 0$, this pair tends to the standard energy-energy flux pair of the non-relativistic Euler system.

More generally, the entropy pairs are determined by

$$\nabla\mathcal{F} \cdot r_j = \lambda_j \nabla\mathcal{U} \cdot r_j, \quad j = 1, 2.$$

Expressing $\mathcal{U}, \mathcal{F}$ as functions of w, z we get

$$\mathcal{F}_w = \lambda_2 \mathcal{U}_w, \qquad \mathcal{F}_z = \lambda_1 \mathcal{U}_z, \tag{4.1}$$

and imply an equation satisfied by the entropy $\mathcal{U} = \mathcal{U}(w, z)$ only:

$$(\lambda_1 \mathcal{U}_z)_w = (\lambda_2 \mathcal{U}_w)_z.$$

That is, $\mathcal{U}$ satisfies

$$\mathcal{U}_{wz} + \frac{\lambda_{2z}}{\lambda_2 - \lambda_1}\mathcal{U}_w - \frac{\lambda_{1w}}{\lambda_2 - \lambda_1}\mathcal{U}_z = 0, \tag{4.2}$$

which we will refer to as the *entropy equation.*

Using the formulas (3.13), the coefficients in (4.2) are found to be

$$\frac{\lambda_{1w}}{\lambda_2 - \lambda_1} = \frac{(1-\varepsilon^2)(1+\varepsilon^2 v)}{4(1-\varepsilon^2 v)}, \qquad \frac{\lambda_{2z}}{\lambda_2 - \lambda_1} = \frac{(1-\varepsilon^2)(1-\varepsilon^2 v)}{4(1+\varepsilon^2 v)},$$

and thus (4.2) becomes

$$\mathcal{U}_{wz} + \overline{b}(w+z)\,\mathcal{U}_w + \overline{a}(w+z)\,\mathcal{U}_z = 0, \tag{4.3}$$

where the coefficients depend on $w + z$ only,

$$
\begin{aligned}
a(v) &:= -\frac{(1-\varepsilon^2)(1+\varepsilon^2 v)}{4(1-\varepsilon^2 v)}, \\
b(v) &:= \frac{(1-\varepsilon^2)(1-\varepsilon^2 v)}{4(1+\varepsilon^2 v)} = -a(-v),
\end{aligned}
\tag{4.4}
$$

in which $v = \overline{v}(w+z)$ and $\overline{a} := a \circ \overline{v}$, $\overline{b} := b \circ \overline{v}$, therefore

$$
\begin{aligned}
\overline{a}(\xi) &= -\frac{1-\varepsilon^2}{4} \cdot \frac{1-\varepsilon+(1+\varepsilon)e^{\varepsilon\xi}}{1+\varepsilon+(1-\varepsilon)e^{\varepsilon\xi}}, \\
\overline{b}(\xi) &= \frac{1-\varepsilon^2}{4} \cdot \frac{1+\varepsilon+(1-\varepsilon)e^{\varepsilon\xi}}{1-\varepsilon+(1+\varepsilon)e^{\varepsilon\xi}} = -\overline{a}(-\xi).
\end{aligned}
\tag{4.5}
$$

We are interested in the solutions to (4.3), which is a linear hyperbolic equation with smooth coefficients. It is well-known that the solutions can be generated from the associated *Riemann function* $\mathcal{R}(w', z'; w, z)$, defined for each fixed (w, z) as the unique solution of the *Goursat problem* for the adjoint operator:

$$
\begin{aligned}
&\mathcal{R}_{w'z'} - \left(\overline{b}(w'+z')\,\mathcal{R}\right)_{w'} - \left(\overline{a}(w'+z')\,\mathcal{R}\right)_{z'} = 0, \\
&\mathcal{R}_{w'}(w', z; w, z) = \overline{a}(w'+z)\,\mathcal{R}(w', z; w, z), \qquad \text{on the line } z' = z, \\
&\mathcal{R}_{z'}(w, z'; w, z) = \overline{b}(w+z')\,\mathcal{R}(w, z'; w, z), \qquad \text{on the line } w' = w.
\end{aligned}
\tag{4.6}
$$

In turn, the Riemann function can be used, for instance, to solve the general characteristic value problem

$$
\begin{aligned}
&\mathcal{U}_{wz} + \overline{b}(w+z)\,\mathcal{U}_w + \overline{a}(w+z)\,\mathcal{U}_z = g, \\
&\mathcal{U}(w, z') = \varphi(w) \qquad \text{on the line } z = z', \\
&\mathcal{U}(w', z) = \psi(z) \qquad \text{on the line } w = w',
\end{aligned}
\tag{4.7}
$$

where φ, ψ are prescribed characteristic data and g is a given source. We have

$$
\begin{aligned}
\mathcal{U}(w, z) := {} & \frac{1}{2}\,\varphi(w)\,\mathcal{R}(w, z'; w, z) + \frac{1}{2}\,\psi(z)\,\mathcal{R}(w', z; w, z) \\
& + \int_{w'}^{w} \frac{1}{2}\,\mathcal{R}(w'', z'; w, z)\,\varphi_w(w'')\,dw'' \\
& + \int_{w'}^{w} \Big((\overline{a}(w''+z')\,\mathcal{R}(w'', z'; w, z) - \frac{1}{2}\,\mathcal{R}_{w'}(w'', z'; w, z) \Big)\,\varphi(w'')\,dw'' \\
& + \int_{z'}^{z} \frac{1}{2}\,\mathcal{R}(w', z''; w, z)\,\psi_z(z'')\,dz'' \\
& + \int_{z'}^{z} \Big((\overline{b}(w'+z'')\,\mathcal{R}(w', z''; w, z) - \frac{1}{2}\,\mathcal{R}_{z'}(w', z''; w, z) \Big)\,\psi(z'')\,dz'' \\
& + \int_{w'}^{w} \int_{z'}^{z} g(w'', z'')\,\mathcal{R}(w'', z''; w, z)\,dw''dz''.
\end{aligned}
\tag{4.8}
$$

4.2. **The non-relativistic limit.** We begin by recalling the construction made in [7] when $\varepsilon = 0$. Taking formally $\varepsilon \to 0$ in (4.3) we get

$$\mathcal{U}^0_{wz} + \frac{1}{4}\left(\mathcal{U}^0_w - \mathcal{U}^0_z\right) = 0.$$

The Riemann function associated with this equation was studied in [7]. Consider the function

$$\mathcal{R}^0(w', z'; w, z) = e^{\{(w-w')-(z-z')\}/4} f_0((w-w')(z-z')), \tag{4.9}$$

where the function $f_0 = f_0(m)$ is (closely related to the Bessel function of order 0) and is defined as the unique solution of the ordinary differential equation

$$m f_0'' + f_0' + f_0/16 = 0, \qquad f_0(0) = 1, \quad f'(0) = -1/16. \tag{4.10}$$

One can check that, for every fixed (w, z), the function $(w', z') \mapsto \mathcal{R}^0(w', z'; w, z)$ defined in (4.9) is the unique solution of the Goursat problem

$$\begin{aligned} &\mathcal{R}^0_{w'z'} - \frac{1}{4}\left(\mathcal{R}^0_{w'} - \mathcal{R}^0_{z'}\right) = 0, \\ &\mathcal{R}^0(w', z; w, z) = e^{(w-w')/4} \quad \text{on the line } z' = z, \\ &\mathcal{R}^0(w, z'; w, z) = e^{-(z-z')/4} \quad \text{on the line } w' = w. \end{aligned} \tag{4.11}$$

We recall from [7] that:

Theorem 4.1 (Entropy kernel of the non-relativistic Euler equations). *Consider the isothermal non-relativistic Euler equations* (1.2) *(with $k = 1$ after normalization). Then the function*

$$\chi^0(w, z) = \begin{cases} \mathcal{R}^0(0, 0; w, z) = e^{(w-z)/4} f_0(wz), & wz < 0, \\ 0, & wz > 0, \end{cases}$$

is a fundamental solution of

$$\chi^0_{wz} + \frac{1}{4}\left(\chi^0_w - \chi^0_z\right) = -2\delta_{w=0} \otimes \delta_{z=0}.$$

It is solely a function of bounded variation and the singularities of its derivatives are given by

$$\begin{aligned} \chi^0_w(w, z) &= e^{-z/4}\Big(-(\operatorname{sgn} z)\,\delta_{w=0} + e^{w/4}\big(\frac{1}{4} f_0(wz) + z f_0'(wz)\big)\, \mathbb{1}_{wz<0}\Big), \\ \chi^0_z(w, z) &= e^{w/4}\Big(-(\operatorname{sgn} w)\,\delta_{z=0} + e^{-z/4}\big(-\frac{1}{4} f_0(wz) + w f_0'(wz)\big)\, \mathbb{1}_{wz<0}\Big), \end{aligned} \tag{4.12}$$

where $\mathbb{1}_{wz<0}$ is the characteristic function associated with the region $wz < 0$.

It is is natural to define the entropy kernel by imposing data on the line $\rho = 1$. (Observe that $w = z = 0$ correspond to $\rho = u = 0$.) Hence, the mathematical entropies of (1.2) that vanish on the vacuum are given by the formula

$$\mathcal{U}(\rho, v) = U(w, z) = \int_{\mathbb{R}} \chi^0(w-s, z-s)\, \psi(s)\, ds, \tag{4.13}$$

valid in each region $\rho < 1$ and $\rho > 1$ (as they avoid the point mass at $w = z = 0$), where $\psi: \mathbb{R} \to \mathbb{R}$ is an arbitrary, integrable function. A similar formula hold for the entropy flux.

In the framework of [7] the Riemann invariants are bounded and this yields the restriction $|v \pm \ln\rho| \leq C$ on the physical variables. In particular, the argument of the function f_0 in the definition of χ^0 remains in a compact set, and the entropy kernel satisfies

$$|\chi^0(\rho, v)| \lesssim \rho^{1/2}.$$

4.3. **Entropy kernel.** We can generalize the results in [7] as follows.

Theorem 4.2 (Entropy kernel of the relativistic Euler equations). *Consider the Euler equations for isothermal, relativistic fluids* (3.1).

1. *The function*

$$\chi^\varepsilon(w, z) = \begin{cases} \mathcal{R}^\varepsilon(0,0;w,z), & w\,z < 0, \\ 0, & w\,z > 0, \end{cases}$$

is a fundamental solution of the entropy equation

$$\chi^\varepsilon_{wz} + \overline{b}(w+z)\,\chi^\varepsilon_w + \overline{a}(w+z)\,\chi^\varepsilon_z = -2\delta_{w=0} \otimes \delta_{z=0}. \tag{4.14}$$

It is solely a function of bounded variation and the singularities of its derivatives are given by

$$\begin{aligned} \chi^\varepsilon_w(w,z) &= \frac{1}{2}\big(1 - \varepsilon + (1+\varepsilon)e^{\varepsilon z}\big)e^{-(1+\varepsilon)^2 z/4}\Big(- (\operatorname{sgn} z)\,\delta_{w=0} + C_1(w,z)\,\mathbb{1}_{wz<0}\Big), \\ \chi^\varepsilon_z(w,z) &= \frac{1}{2}\big(1 + \varepsilon + (1-\varepsilon)e^{\varepsilon w}\big)\,e^{(1-\varepsilon)^2 w/4}\,\Big(- (\operatorname{sgn} w)\,\delta_{z=0} + C_2(w,z)\,\mathbb{1}_{wz<0}\Big), \end{aligned} \tag{4.15}$$

where $\mathbb{1}_{wz<0}$ is the characteristic function associated with the region $wz < 0$ and C_1, C_2 are smooth functions, with

$$\begin{aligned} C_1(0,z) &= \frac{1-\varepsilon^2}{4}\big(1 - (1-\varepsilon^2)z/4\big), \\ C_2(w,0) &= -\frac{1-\varepsilon^2}{4}\big(1 + (1-\varepsilon^2)w/4\big). \end{aligned}$$

2. *In the physical variables ρ, v, the kernel $\chi = \chi(\rho, v)$ has compact support in the variable $v \in (-1/\varepsilon, 1/\varepsilon)$ (for every fixed value ρ), and satisfies*

$$\lim_{\rho\to 1}\chi^\varepsilon(\rho,\cdot) = 0, \qquad \lim_{\rho\to\pm 1}\chi^\varepsilon_\rho(\rho,\cdot) = \pm\delta_{u=0} \quad \textit{in the weak sense,}$$

$$|\chi(\rho,u)| \lesssim \rho^\alpha, \qquad \alpha := \frac{(1-\varepsilon)^2}{2(1+\varepsilon^2)}.$$

Observe that, as $\varepsilon \to 0$, the expansion (4.15) converges to the expansion known in the non-relativistic case (4.12).

5. Sketch of the proofs

In this section we sketch the proofs, especially the one of Theorem 4.2.

Lemma 5.1. *The traces of χ along its support $(wz < 0)$ are given by*

$$\chi(w,z) \to \begin{cases} \frac{1}{2}\big(1+\varepsilon+(1-\varepsilon)e^{\varepsilon w}\big)\, e^{(1-\varepsilon)^2 w/4}, & z \to 0, \\ \frac{1}{2}\big(1-\varepsilon+(1+\varepsilon)e^{\varepsilon z}\big)\, e^{-(1+\varepsilon)^2 z/4}, & w \to 0, \end{cases} \tag{5.1}$$

and moreover on the boundary $wz = 0$ we have

$$\chi(\rho, v) \lesssim \rho^\alpha,$$

with α defined in Theorem 4.2.

Proof. We need to integrate out the boundary conditions arising in 4.6. The second differential equation in (4.6) implies

$$\ln|\mathcal{R}(w',z;w,z)| + const.(w,z) = -\frac{1-\varepsilon^2}{4}\Big(-(w'+z) + 2\int^{w'} \frac{1}{1-\varepsilon^2 v}dw'\Big)$$

where

$$\begin{aligned} \int^{w'} \frac{1}{1-\varepsilon^2 v}dw' &= \int^{w'} \frac{1}{1-\varepsilon\tanh\left(\varepsilon(w'+z)/2\right)}dw' \\ &= \frac{2}{1-\varepsilon^2}\ln\Big\{(1-\varepsilon)e^{\varepsilon(w'+z)} + 1 + \varepsilon\Big\} + \frac{1}{1+\varepsilon}(w'+z). \end{aligned}$$

The third equation in (4.6) implies

$$\ln|\mathcal{R}(w,z';w,z)| + const.(w,z) = \frac{1-\varepsilon^2}{4}\Big(-(w+z') + 2\int^{z'} \frac{1}{1+\varepsilon^2 v}dz'\Big),$$

where

$$\begin{aligned} \int^{z'} \frac{1}{1+\varepsilon^2 v}dz' &= \int^{z'} \frac{1}{1+\varepsilon\tanh\left(\varepsilon(w+z')/2\right)}dz' \\ &= -\frac{2}{1-\varepsilon^2}\ln\Big\{(1+\varepsilon)e^{\varepsilon(w+z')} + 1 - \varepsilon\Big\} + \frac{1}{1-\varepsilon}(w+z'). \end{aligned}$$

Therefore, since $\mathcal{R}(w,z;w,z) = 1$, we find

$$\mathcal{R}(w',z';w,z) = \begin{cases} e^{-(1-\varepsilon)^2(w'-w)/4} & \\ \quad\times\dfrac{1-\varepsilon+(1+\varepsilon)e^{-\varepsilon(w+z)}}{(1-\varepsilon)e^{\varepsilon(w'-w)}+(1+\varepsilon)e^{-\varepsilon(w+z)}} =: A(w';w,z), & \text{at } z' = z, \\ e^{(1+\varepsilon)^2(z'-z)/4} & \\ \quad\times\dfrac{1+\varepsilon+(1-\varepsilon)e^{-\varepsilon(w+z)}}{(1+\varepsilon)e^{\varepsilon(z'-z)}+(1-\varepsilon)e^{-\varepsilon(w+z)}} =: B(z';w,z), & \text{at } w' = w. \end{cases} \tag{5.2}$$

In particular,

$$\mathcal{R}(0,0;w,z)=\begin{cases} e^{(1-\varepsilon)^2w/4}\cdot\dfrac{1+\varepsilon+(1-\varepsilon)e^{\varepsilon w}}{2}=A(0;w,0)=:A(w), & \text{at } z=0,\\ e^{-(1+\varepsilon)^2z/4}\cdot\dfrac{1-\varepsilon+(1+\varepsilon)e^{\varepsilon z}}{2}=B(0;0,z)=:B(z), & \text{at } w=0,\end{cases} \tag{5.3}$$

which provides the desired behavior on χ.

Next, to estimate χ we write along the boundary $w=0$ (thus $u=-R$ and $z=-2R$)

$$B(z)=\frac{1}{2}\rho^{\frac{(1+\varepsilon)^2}{2(1+\varepsilon^2)}}\left(1-\varepsilon+(1+\varepsilon)\rho^{-\frac{2\varepsilon}{1+\varepsilon^2}}\right),$$

which shows that $\chi(\rho,v)\lesssim\rho^{\alpha}$. On the boundary $z=0$ we have $u=R$ and $w=-2R$ and we find a stronger estimate $A(w)\lesssim\rho^{\alpha'}$ with $\alpha'=\dfrac{(1+\varepsilon)^2}{2(1+\varepsilon^2)}$. □

Lemma 5.2. *The entropy kernel $\chi(w,z)=\mathcal{R}(0,0;w,z)\mathbb{1}_{wz<0}$ is a fundamental solution of the entropy equation.*

Proof. We obtain

$$\chi_w(w,z)=\mathcal{R}_w(0,0;w,z)\mathbb{1}_{wz<0}-(\operatorname{sgn} z)B(0,0;0,z)\delta_{w=0},$$
$$\chi_z(w,z)=\mathcal{R}_z(0,0;w,z)\mathbb{1}_{wz<0}-(\operatorname{sgn} w)A(0,0;w,0)\delta_{z=0}$$

and

$$\begin{aligned}\chi_{wz}(w,z)&=\mathcal{R}_{wz}(0,0;w,z)\mathbb{1}_{wz<0}-(\operatorname{sgn} w)\mathcal{R}_w(0,0;w,0)\delta_{z=0}\\&\quad-(\operatorname{sgn} z)\partial_z B(0,0;0,z)\delta_{w=0}-2B(0,0;0,0)\delta_{w=0}\otimes\delta_{z=0}\\&=\mathcal{R}_{wz}(0,0;w,z)\mathbb{1}_{wz<0}-(\operatorname{sgn} w)A_w(0,0;w,0)\delta_{z=0}\\&\quad-(\operatorname{sgn} z)\partial_z B(0,0;0,z)\delta_{w=0}-2\delta_{w=0}\otimes\delta_{z=0} &(5.4)\\&=\mathcal{R}_{wz}(0,0;w,z)\mathbb{1}_{wz<0}+(\operatorname{sgn} w)\overline{a}(w)A(0,0;w,0)\delta_{z=0}\\&\quad+(\operatorname{sgn} z)\overline{b}(z)(0,0;0,z)\delta_{w=0}-2\delta_{w=0}\otimes\delta_{z=0}. &(5.5)\end{aligned}$$

Therefore we have

$$\begin{aligned}&\chi_{wz}+\overline{b}(w+z)\chi_w+\overline{a}(w+z)\chi_z\\&=\mathcal{R}_{wz}(0,0;w,z)\mathbb{1}_{wz<0}+(\operatorname{sgn} w)\overline{a}(w)A(0,0;w,0)\delta_{z=0}\\&\quad+(\operatorname{sgn} z)\overline{b}(z)(0,0;0,z)\delta_{w=0}-2\delta_{w=0}\otimes\delta_{z=0}\\&\quad+\overline{b}(w+z)\left\{\mathcal{R}_w(0,0;w,z)\mathbb{1}_{wz<0}-(\operatorname{sgn} z)B(0,0;0,z)\delta_{w=0}\right\}\\&\quad+\overline{a}(w+z)\left\{\mathcal{R}_z(0,0;w,z)\mathbb{1}_{wz<0}-(\operatorname{sgn} w)A(0,0;w,0)\delta_{z=0}\right\}\\&=\left\{\mathcal{R}_{wz}(0,0;w,z)+\overline{b}(w+z)\mathcal{R}_w(0,0;w,z)+\overline{a}(w+z)\mathcal{R}_z(0,0;w,z)\right\}\mathbb{1}_{wz<0}\\&\quad+(\operatorname{sgn} w)\left\{\overline{a}(w)-\overline{a}(w+z)\right\}A(0,0;w,0)\delta_{z=0}\\&\quad+(\operatorname{sgn} z)\left\{\overline{b}(z)-\overline{b}(w+z)\right\}B(0,0;0,z)\delta_{w=0}-2\delta_{w=0}\otimes\delta_{z=0}\\&=-2\delta_{w=0}\otimes\delta_{z=0}\,.\end{aligned}$$

□

Lemma 5.3. *The traces of the derivatives χ_w and χ_z along the boundaries $z = 0$ and $w = 0$ (while keeping $wz < 0$) of the support of χ are given as follows:*

$$\chi_w(0,z) = \frac{1-\varepsilon^2}{8}\big(1-(1-\varepsilon^2)z/4\big)\big(1-\varepsilon+(1+\varepsilon)e^{\varepsilon z}\big)e^{-(1+\varepsilon)^2 z/4}$$
$$\chi_z(w,0) = -\frac{1-\varepsilon^2}{8}\big(1+(1-\varepsilon^2)w/4\big)\big(1+\varepsilon+(1-\varepsilon)e^{\varepsilon w}\big)\, e^{(1-\varepsilon)^2 w/4}.$$

Proof. We give the proof for the boundary $z = 0-$ with $w > 0$, the calculation for the other boundaries being similar. Using (4.7) and the expressions (4.5) we write

$$\frac{d}{dw}\chi_z(w,0-) + \overline{a}(w)\,\chi_z(w,0-) = -\overline{b}(w)\,\chi_w(w,0-),$$

where the coefficients a and b can be regarded as functions of $w+z = w$ and the right-hand side is already known. By integrating out the above equation we obtain

$$\chi_z(w,0-) = -\int_0^w \overline{b}(w')\,\chi_w(w',0)\exp\left(\int_w^{w'}\overline{a}(w'')\,dw''\right)dw' + \chi_z(0,0)\exp\left(-\int_0^w \overline{a}(w')\right), \tag{5.6}$$

where here $\chi_z(0,0)$ stands for $\lim_{w\to 0+}\chi_z(w,0-)$.

On the other hand, from (5.1) we determine the derivatives

$$\chi_w(w,0) := \frac{d}{dw}\left(\frac{1}{2}\big(1+\varepsilon+(1-\varepsilon)e^{\varepsilon w}\big)\, e^{(1-\varepsilon)^2 w/4}\right) = \frac{1-\varepsilon^2}{8}\big(1-\varepsilon+(1+\varepsilon)e^{\varepsilon w}\big)\, e^{(1-\varepsilon)^2 w/4}\Big)$$

and

$$\chi_z(0,z) := \frac{d}{dz}\left(\frac{1}{2}\big(1-\varepsilon+(1+\varepsilon)e^{\varepsilon z}\big)\, e^{-(1+\varepsilon)^2 z/4}\right) = -\frac{1-\varepsilon^2}{8}\big(1+\varepsilon+(1-\varepsilon)e^{\varepsilon z}\big)\, e^{-(1+\varepsilon)^2 z/4}\Big).$$

In particular, $\chi_z(0,0) = -(1-\varepsilon^2)/4$. This allows us to compute

$$\exp\left(-\int_0^w \overline{a}(w')\,dw'\right) = \frac{1}{2}\big(1+\varepsilon+(1-\varepsilon)e^{\varepsilon w}\big)\, e^{(1-\varepsilon)^2 w/4}.$$

and

$$\exp\left(\int_w^{w'}\overline{a}(w'')\,dw''\right) = \frac{1+\varepsilon+(1-\varepsilon)e^{\varepsilon w}}{1+\varepsilon+(1-\varepsilon)e^{\varepsilon w'}}\, e^{-(1-\varepsilon)^2(w'-w)/4}.$$

Returning to (5.6), we find

$$\overline{b}(w')\,\chi_w(w',0)\exp\left(\int_w^{w'}\overline{a}(w'')\,dw''\right) = \frac{(1-\varepsilon^2)^2}{32}\big(1+\varepsilon+(1-\varepsilon)e^{\varepsilon w}\big)\,\exp\big((1-\varepsilon)^2 w/4\big),$$

and we arrive at to the result. □

For further details on the proof, especially the uniform estimates for the Lax-Friedrichs scheme and the reduction of the Young measure, we refer the reader to [8].

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